On the exterior Dirichlet problem for Hessian equations

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Abstract

In this paper, we establish a theorem on the existence of the solutions of the exterior Dirichlet problem for Hessian equations with prescribed asymptotic behavior at infinity. This extends a result of Caffarelli and Li in [3] for the Monge-Ampère equation to Hessian equations.

1 Introduction

In this paper, we consider the solvability of the Dirichlet problem for Hessian equations

$$\sigma_k(\lambda(D^2u)) = 1 \tag{1.1}$$

on exterior domains $\mathbb{R}^n \setminus D$, where D is a bounded open set in $\mathbb{R}^n$, $n \geq 3$, $\lambda(D^2u)$ denotes the eigenvalues $\lambda_1, \cdots, \lambda_n$ of the Hessian matrix of u. Here

$$\sigma_k(\lambda) = \sum_{1\leq i_1<\cdots<i_k\leq n} \lambda_{i_1}\cdots\lambda_{i_k}$$

is the k-th elementary symmetric function of n variations, $k = 1, \cdots, n$. Note that the case $k = 1$ corresponds to the Poisson's equation, which is a linear equation. There have been extensive literatures on the exterior Dirichlet problem for linear elliptic equations of second order, see [19] and the references therein. For $2 \leq k \leq n$, the Hessian equation (1.1) is an important class of fully nonlinear elliptic equations. Especially, for $k = n$, we have the Monge-Ampère equation $\det(D^2u) = 1$.

For the Monge-Ampère equation, a classical theorem of Jörgens ([17]), Calabi ([5]), and Pogorelov ([20]) states that any classical convex solution of $\det(D^2u) = 1$ in $\mathbb{R}^n$ must be a quadratic polynomial. A simpler and more analytic proof, along the lines of affine geometry, was later given by Cheng and Yau [6]. Caffarelli [1] extended the result for classical solutions to viscosity solutions. Another proof of this theorem was given by Jost and Yin in [18]. Trudinger and Wang [24] proved that if Ω is an

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open convex subset of $\mathbb{R}^n$ and u is a convex C^2 solution of $\det(D^2u) = 1$ in Ω with $\lim_{x\to\partial\Omega} u(x) = \infty$, then $\Omega = \mathbb{R}^n$ and u is quadratic.

Caffarelli and the third author [3] extended the Jörgens-Calabi-Pogorelov theorem to exterior domains. They proved that if u is a convex viscosity solution of $\det(D^2u) = 1$ outside a bounded subset of $\mathbb{R}^n$, $n \geq 3$, then there exist a $n\times n$ real symmetric positive definite matrix A, a vector $b \in \mathbb{R}^n$, and a constant $c \in \mathbb{R}$ such that

$$\limsup_{|x|\to\infty} \left(|x|^{n-2} \left| u(x) - \left(\frac{1}{2} x^T Ax + b \cdot x + c \right) \right| \right) < \infty. \tag{1.2}$$

With this prescribed asymptotic behavior at infinity, an existence result for the exterior Dirichlet problem for the Monge-Ampère equation in $\mathbb{R}^n$, $n \geq 3$, was also established in [3]. In this paper, we will extend the existence theorem to the Dirichlet problem for Hessian equations (1.1) with $2 \leq k \leq n-1$ on exterior domains, with an appropriate asymptotic behavior at infinity. In dimension two, similar problems were studied by Ferrer, Martínez and Milán in [12, 13] using complex variable method. See also Delanoë [11].

We remark that for the case that $A = c^* I$, where

$$c^* = (C_n^k)^{-1/k}, \quad C_n^k = \frac{n!}{(n-k)!k!},$$

I is the $n \times n$ identity matrix and $1 \leq k \leq n$, the exterior Dirichlet problem of Hessian equation (1.1) has been investigated in [9, 10]. For interior domains, there have been many well known results on the solvability of Hessian equations. For instance, Caffarelli, Nirenberg and Spruck [4] established the classical solvability of the Dirichlet problem, Trudinger [23] proved the existence and uniqueness of weak solutions, and Urbas [25] demonstrated the existence of viscosity solutions. Jian [16] studied the Hessian equations with infinite Dirichlet boundary value conditions.

For readers' convenience, we recall the definition of viscosity solutions to Hessian equations (see [2, 25] and the references therein). We say that a function $u \in C^2(\mathbb{R}^n \setminus \overline{D})$ is admissible (or k-convex) if $\lambda(D^2u) \in \overline{\Gamma}_k$ in $\mathbb{R}^n \setminus \overline{D}$, where Γ_k is the connected component of $\{\lambda \in \mathbb{R}^n \mid \sigma_k(\lambda) > 0\}$ containing

$$\Gamma^+ = \{\lambda \in \mathbb{R}^n \mid \lambda_i > 0, i = 1, \cdots, n\}.$$

It is well known that Γ_k is a convex symmetric cone with vertex at the origin. Moreover,

$$\Gamma_k = \{\lambda \in \mathbb{R}^n \mid \sigma_j(\lambda) > 0, \text{ for all } j = 1, \cdots, k\}.$$

See [4, 22]. Clearly, $\Gamma_k \subseteq \Gamma_j$ for $k \geq j$, and Γ_1 is the half space $\{\lambda \in \mathbb{R}^n \mid \lambda_1 + \cdots + \lambda_n > 0\}$, while $\Gamma_n = \Gamma^+$. We use the following definitions, which can be found in [21].

Let $\Omega \subset \mathbb{R}^n$, we use $\mathrm{USC}(\Omega)$ and $\mathrm{LSC}(\Omega)$ to denote respectively the set of upper and lower semicontinuous real valued functions on Ω.

Definition 1.1. A function $u \in \mathrm{USC}(\mathbb{R}^n \setminus \overline{D})$ is said to be a viscosity subsolution of equation (1.1) in $\mathbb{R}^n \setminus \overline{D}$ (or say that u satisfies $\sigma_k(\lambda(D^2u)) \geq 1$ in $\mathbb{R}^n \setminus \overline{D}$ in the viscosity sense), if for any function $\psi \in C^2(\mathbb{R}^n \setminus \overline{D})$ and point $\bar{x} \in \mathbb{R}^n \setminus \overline{D}$ satisfying

$$\psi(\bar{x}) = u(\bar{x}) \quad \text{and} \quad \psi \geq u \text{ on } \mathbb{R}^n \setminus \overline{D},$$

we have

$$\sigma_k(\lambda(D^2\psi(\overline{x}))) \geq 1.$$

A function $u \in \mathrm{LSC}(\mathbb{R}^n \setminus \overline{D})$ is said to be a viscosity supersolution of (1.1) in $\mathbb{R}^n \setminus \overline{D}$ (or say that u satisfies $\sigma_k(\lambda(D^2u)) \leq 1$ in $\mathbb{R}^n \setminus \overline{D}$ in the viscosity sense), if for any k-convex function $\psi \in C^2(\mathbb{R}^n \setminus \overline{D})$ and point $\overline{x} \in \mathbb{R}^n \setminus \overline{D}$ satisfying

$$\psi(\overline{x}) = u(\overline{x}) \quad \text{and} \quad \psi \leq u \text{ on } \mathbb{R}^n \setminus \overline{D},$$

we have

$$\sigma_k(\lambda(D^2\psi(\overline{x}))) \leq 1.$$

A function $u \in C^0(\mathbb{R}^n \setminus \overline{D})$ is said to be a viscosity solution of (1.1), if it is both a viscosity subsolution and supersolution of (1.1).

It is well known that a function $u \in C^2(\mathbb{R}^n \setminus \overline{D})$ is a viscosity solution (respectively, subsolution, supersolution) of (1.1) if and only if it is a k-convex classical solution (respectively, subsolution, supersolution).

Definition 1.2. Let $\varphi \in C^0(\partial D)$. A function $u \in \mathrm{USC}(\mathbb{R}^n \setminus D)$ ($u \in \mathrm{LSC}(\mathbb{R}^n \setminus D)$) is said to be a viscosity subsolution (supersolution) of the Dirichlet problem

$$\begin{cases} \sigma_k(\lambda(D^2u)) = 1, & \text{in } \mathbb{R}^n \setminus \overline{D}, \\ u = \varphi, & \text{on } \partial D, \end{cases} \tag{1.3}$$

if u is a viscosity subsolution (supersolution) of (1.1) in $\mathbb{R}^n \setminus \overline{D}$ and $u \leq (\geq)\ \varphi$ on ∂D. A function $u \in C^0(\mathbb{R}^n \setminus D)$ is said to be a viscosity solution of (1.3) if it is both a subsolution and a supersolution.

Let

$$\mathcal{A}_k = \left\{ A \,\middle|\, A \text{ is a real } n \times n \text{ symmetric positive definite matrix, with } \sigma_k(\lambda(A)) = 1 \right\}.$$

Our main result is

Theorem 1.1. *Let D be a smooth, bounded, strictly convex open subset of $\mathbb{R}^n$, $n \geq 3$, and let $\varphi \in C^2(\partial D)$. Then for any given $b \in \mathbb{R}^n$ and any given $A \in \mathcal{A}_k$ with $2 \leq k \leq n$, there exists some constant c_*, depending only on n, b, A, D and $||\varphi||_{C^2(\partial D)}$, such that for every $c > c_*$ there exists a unique viscosity solution $u \in C^0(\mathbb{R}^n \setminus D)$ of* (1.3) *and*

$$\limsup_{|x|\to\infty} \left(|x|^{\theta(n-2)} \left| u(x) - \left(\frac{1}{2} x^T A x + b \cdot x + c \right) \right| \right) < \infty, \tag{1.4}$$

where $\theta \in \left[\frac{k-2}{n-2}, 1\right]$ is a constant depending only on n, k, and A.

Remark 1.1. For the two cases (i) $k = n$, the Monge-Ampère equations with any $A \in \mathcal{A}_n$; and (ii) $2 \leq k \leq n-1$, (1.4) with $A = c^* I \in \mathcal{A}_k$, Theorem 1.1 has been proved by Caffarelli-Li [3] and Dai-Bao [10], respectively, where $\theta = 1$. Moreover, for the symmetric case $A = c^* I$, Wang-Bao [26] have proved that for $2 \leq k \leq n$, there exists a $\overline{c}(k, n)$ such that there is no classical radial solution of (1.3) and (1.4) if $c < \overline{c}(k, n)$.

Recall that any real symmetric matrix A has an eigen-decomposition $A = O^T \Lambda O$ where O is an orthogonal matrix, and Λ is a diagonal matrix. That is, A may be regarded as a real diagonal matrix Λ that has been re-expressed in some new coordinate system, and the eigenvalues $\lambda(A) = \lambda(\Lambda)$. Let

$$y = Ox, \quad \text{and} \quad v(y) = u(O^{-1}y),$$

then (1.3) and (1.4) become

$$\begin{cases} \sigma_k(\lambda(D_y^2 v)) = 1, & \text{in } \mathbb{R}^n \setminus \overline{\widetilde{D}}, \\ v = \varphi(O^{-1}y), & \text{on } \partial\widetilde{D} \end{cases}$$

and

$$\limsup_{|y|\to\infty} \left(|O^{-1}y|^{\theta(n-2)} \left| v(y) - \left(\frac{1}{2} y^T \Lambda y + bO^{-1} \cdot y + c\right) \right| \right) < \infty,$$

where $\widetilde{D}$ is transformed from D under $y = Ox$. So, without loss of generality, we always assume that A is diagonal in this paper.

If A is diagonal and $A \in \mathcal{A}_n$, then $\sigma_n(\lambda(A)) = 1$, and we can find a diagonal matrix Q with $\det Q = 1$ such that $QAQ = I \in \mathcal{A}_n$. Clearly, $\lambda(I)$ is not necessarily the same as $\lambda(A)$, but under transformation $y = Qx$, we still have

$$\det\left(D_x^2 u\right) = \det\left(QD_y^2 uQ\right) = \det\left(D_y^2 u\right).$$

Therefore, when the Monge-Ampère equation is considered, Caffarelli and Li [3] can assume without loss of generality that $A = I$. However, when $2 \le k \le n-1$, if A is diagonal and $A \in \mathcal{A}_k$, $\sigma_k(\lambda(A)) = 1$, although we can also find a diagonal matrix Q such that $QAQ = c^* I \in \mathcal{A}_k$, it is clear that $\lambda(A) \neq \lambda(c^* I)$ unless $A = c^* I$, and for Hessian operator

$$\sigma_k\left(\lambda(QD_y^2 uQ)\right) \neq \sigma_k(\lambda(Q))\sigma_k\left(\lambda(D_y^2 u)\right)\sigma_k(\lambda(Q)).$$

So, in order to prove Theorem 1.1, we are only allowed to assume that A is diagonal, but we can not further assume that $A = c^* I$.

Definition 1.3. For a diagonal matrix $A = \operatorname{diag}(a_1, a_2, \cdots, a_n)$, we call u a generalized symmetric function with respect to A, if u is a function of

$$s = \frac{1}{2} x^T A x = \frac{1}{2} \sum_{i=1}^n a_i x_i^2.$$

If u is a generalized symmetric function with respect to A and u is a solution (respectively, subsolution, supersolution) of the Hessian equation (1.1), then we call u a generalized symmetric solution (respectively, subsolution, supersolution) of (1.1).

In this paper we often abuse notations slightly by writing $u(x) = u(\frac{1}{2} x^T A x)$ for a generalized symmetric function with respect to A. Clearly, for diagonal matrix $A = \operatorname{diag}(a_1, a_2, \cdots, a_n) \in \mathcal{A}_k$, and real constants μ_1, μ_2, with $\mu_1^k = 1$,

$$\omega(s) = \mu_1 s + \mu_2, \quad s = \frac{1}{2} \sum_{i=1}^n a_i x_i^2 \tag{1.5}$$

satisfies the Hessian equation (1.1) and $\omega''(s) \equiv 0$.

First, we will derive a formula of $\sigma_k(\lambda(M))$ for matrices M of the form

$$M = \left(p_i\delta_{ij} - \beta\, q_i q_j\right)_{n\times n}, \tag{1.6}$$

where $p = (p_1, p_2, \cdots, p_n)$, $q = (q_1, q_2, \cdots, q_n)$ and $\beta \in \mathbb{R}$.

Proposition 1.2. *If M is a $n \times n$ matrix of the form* (1.6) *for $p = (p_1, p_2, \cdots, p_n)$, $q = (q_1, q_2, \cdots, q_n)$ and $\beta \in \mathbb{R}$, then we have*

$$\sigma_k(\lambda(M)) = \sigma_k(p) - \beta \sum_{i=1}^n q_i^2 \sigma_{k-1;i}(p), \tag{1.7}$$

where $\sigma_{k-1;i}(p) = \sigma_{k-1}(p)|_{p_i=0}$.

For any $A = \mathrm{diag}(a_1, a_2, \cdots, a_n)$, suppose $\omega \in C^2(\mathbb{R}^n)$ is a generalized symmetric function with respect to A, that is,

$$\omega(x) = \omega\left(\frac{1}{2}\sum_{i=1}^n a_i x_i^2\right),$$

then

$$D_i\omega(x) = \omega'(s)a_i x_i,$$
$$D_{ij}\omega(x) = \omega'(s)a_i\delta_{ij} + \omega''(s)(a_i x_i)(a_j x_j).$$

We have the following lemma.

Lemma 1.3. *For any $A = \mathrm{diag}(a_1, a_2, \cdots, a_n)$, if $\omega \in C^2(\mathbb{R}^n)$ is a generalized symmetric function with respect to A, then, with $a = (a_1, a_2, \cdots, a_n)$,*

$$\sigma_k(\lambda(D^2\omega)) = \sigma_k(a)(\omega')^k + \omega''(\omega')^{k-1}\sum_{i=1}^n \sigma_{k-1;i}(a)(a_i x_i)^2. \tag{1.8}$$

If $A = c^* I$, $2 \le k \le n$, then there exist a family of radially symmetric functions

$$\overline{\omega}_k(s) = \int_1^s \left(1 + \alpha\, t^{-\frac{n}{2}}\right)^{\frac{1}{k}} dt, \quad \alpha > 0,\ s > 0,$$

satisfying

$$\sigma_k(\lambda(D^2\omega)) = 1, \quad \text{in } \mathbb{R}^n \setminus \{0\}.$$

Such radially symmetric solutions play an important role to the solvability of the exterior Dirichlet problems studied by Caffarelli-Li [3] and by Dai-Bao [10]. However, for any given $A \in \mathcal{A}_k$ with $2 \le k \le n-1$, it is not enough to prove Theorem 1.1 only using these radially symmetric functions. Due to the invariance of (1.1) for $k = n$, the Monge-Ampère equation, under affine transformations, $\overline{\omega}_n(\frac{1}{2}x^T Ax)$ is a solution of (1.1) in $\mathbb{R}^n \setminus \{0\}$ for $A \in \mathcal{A}_n$. So the Monge-Ampère equation has generalized symmetric solutions with respect to A for every $A \in \mathcal{A}_n$. A natural question is that whether (1.1) with $2 \le k \le n-1$ has generalized symmetric solutions with respect to A for every $A \in \mathcal{A}_k$ besides those of the form (1.5).

For this, we have

Proposition 1.4. *For $A = \mathrm{diag}(a_1, a_2, \cdots, a_n) \in \mathcal{A}_k$, $1 \le k \le n$, and $0 < \alpha < \beta < \infty$, if there exists an $\omega \in C^2(\alpha,\beta)$ with $\omega'' \not\equiv 0$ in (α,β), such that $\omega(x) = \omega(\frac{1}{2}\sum_{i=1}^n a_i x_i^2)$ is a generalized symmetric solution of the Hessian equation* (1.1) *in $\{x \in \mathbb{R}^n \mid \alpha < \frac{1}{2}\sum_{i=1}^n a_i x_i^2 < \beta\}$, then*

$$k = n \quad \text{or} \quad a_1 = a_2 = \cdots = a_n = c^*,$$

where $c^* = (C_n^k)^{-1/k}$, $C_n^k = \frac{n!}{(n-k)!k!}$, and vice versa.

This means that for $A = \mathrm{diag}(a_1, a_2, \cdots, a_n) \in \mathcal{A}_k$, $2 \le k \le n-1$, $\omega(\frac{1}{2}x^T Ax)$ is in general not a solution of (1.1).

To prove Theorem 1.1 for $2 \le k \le n-1$, it suffices to obtain enough subsolutions with appropriate properties. We construct such subsolutions which are generalized symmetric functions with respect to A. This is the main new ingredient in our proof of the theorem.

This paper is set out as follows. In the next section we construct a family of generalized symmetric smooth k-convex subsolutions of (1.1) in $\mathbb{R}^n \setminus \{0\}$. In Section 3, we prove Theorem 1.1 using Perron's method.

2 Generalized symmetric solutions and subsolutions

In this section, we first derive formula (1.7) and (1.8), then prove Proposition 1.4, and finally construct a family of generalized symmetric smooth k-convex subsolutions of (1.1).

For $A = \mathrm{diag}(a_1, a_2 \cdots, a_n)$, we denote $\lambda(A) = (a_1, a_2 \cdots, a_n) := a$. If $A \in \mathcal{A}_k$, then we have $a_i > 0$ $(i = 1, 2, \cdots, n)$ and $\sigma_k(a) = 1$. Here we introduce some notations. For any fixed t-tuple $\{i_1, \cdots, i_t\}$, $1 \le t \le n-k$, we define

$$\sigma_{k;i_1\cdots i_t}(a) = \sigma_k(a)|_{a_{i_1}=\cdots=a_{i_t}=0},$$

that is, $\sigma_{k;i_1\cdots i_t}$ is the k-th order elementary symmetric function of the $n-t$ variables $\left\{a_i \,\middle|\, i \in \{1, 2, \cdots, n\} \setminus \{i_1, i_2, \cdots, i_t\}\right\}$. The following properties of the functions σ_k will be used in this paper:

$$\sigma_k(a) = \sigma_{k;i}(a) + a_i\sigma_{k-1;i}(a), \qquad i = 1, 2, \cdots, n, \tag{2.1}$$

and

$$\sum_{i=1}^n a_i\sigma_{k-1;i}(a) = k\sigma_k(a). \tag{2.2}$$

Now we prove Proposition 1.2 to derive a formula of $\sigma_k(\lambda(M))$ for matrices M of the form (1.6).

Proof of Proposition 1.2. If $\beta = 0$, (1.7) is obvious. If $\beta \neq 0$, we work with

$$\widehat{M} = \frac{1}{\beta}M = (\hat{p}_i\delta_{ij} - q_iq_j), \qquad \hat{p} = \frac{p}{\beta}.$$

Therefore we only need to prove Proposition 1.2 for $\beta = 1$, which we assume in the rest of the proof.

Denote

$$D_n\left(\{p_1, p_2, \cdots, p_n\}; \{q_1, q_2, \cdots, q_n\}; \lambda\right) := \det(\lambda I - M). \tag{2.3}$$

By direct computations, we have

$$\begin{aligned}
&D_n\left(\{p_1, p_2, \cdots, p_n\}; \{q_1, q_2, \cdots, q_n\}; \lambda\right)\\
&= \begin{vmatrix} \lambda - p_1 + q_1^2 & q_1q_2 & \cdots & q_1q_{n-1} & q_1q_n \\ q_2q_1 & \lambda - p_2 + q_2^2 & \cdots & q_2q_{n-1} & q_2q_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ q_{n-1}q_1 & q_{n-1}q_2 & \cdots & \lambda - p_{n-1} + q_{n-1}^2 & q_{n-1}q_n \\ q_nq_1 & q_nq_2 & \cdots & q_nq_{n-1} & \lambda - p_n + q_n^2 \end{vmatrix}\\
&= \begin{vmatrix} \lambda - p_1 + q_1^2 & q_1q_2 & \cdots & q_1q_{n-1} & 0 \\ q_2q_1 & \lambda - p_2 + q_2^2 & \cdots & q_2q_{n-1} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ q_{n-1}q_1 & q_{n-1}q_2 & \cdots & \lambda - p_{n-1} + q_{n-1}^2 & 0 \\ q_nq_1 & q_nq_2 & \cdots & q_nq_{n-1} & \lambda - p_n \end{vmatrix}\\
&\quad + \begin{vmatrix} \lambda - p_1 + q_1^2 & q_1q_2 & \cdots & q_1q_{n-1} & q_1q_n \\ q_2q_1 & \lambda - p_2 + q_2^2 & \cdots & q_2q_{n-1} & q_2q_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ q_{n-1}q_1 & q_{n-1}q_2 & \cdots & \lambda - p_{n-1} + q_{n-1}^2 & q_{n-1}q_n \\ q_nq_1 & q_nq_2 & \cdots & q_nq_{n-1} & q_n^2 \end{vmatrix}\\
&= (\lambda - p_n)D_{n-1}\left(\{p_1, p_2, \cdots, p_{n-1}\}; \{q_1, q_2, \cdots, q_{n-1}\}; \lambda\right)\\
&\quad + q_n \begin{vmatrix} \lambda - p_1 + q_1^2 & q_1q_2 & \cdots & q_1q_{n-1} & q_1q_n \\ q_2q_1 & \lambda - p_2 + q_2^2 & \cdots & q_2q_{n-1} & q_2q_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ q_{n-1}q_1 & q_{n-1}q_2 & \cdots & \lambda - p_{n-1} + q_{n-1}^2 & q_{n-1}q_n \\ q_1 & q_2 & \cdots & q_{n-1} & q_n \end{vmatrix}.
\end{aligned}$$

For the second term, multiplying its last row by $-q_i$ $(i \neq n)$ and adding to the i_{th} row, respectively, we obtain

$$\begin{aligned}
&\begin{vmatrix} \lambda - p_1 + q_1^2 & q_1q_2 & \cdots & q_1q_{n-1} & q_1q_n \\ q_2q_1 & \lambda - p_2 + q_2^2 & \cdots & q_2q_{n-1} & q_2q_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ q_{n-1}q_1 & q_{n-1}q_2 & \cdots & \lambda - p_{n-1} + q_{n-1}^2 & q_{n-1}q_n \\ q_1 & q_2 & \cdots & q_{n-1} & q_n \end{vmatrix}\\
&= \begin{vmatrix} \lambda - p_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda - p_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda - p_{n-1} & 0 \\ q_1 & q_2 & \cdots & q_{n-1} & q_n \end{vmatrix}\\
&= q_n(\lambda - p_1)(\lambda - p_2)\cdots(\lambda - p_{n-1}).
\end{aligned}$$

Hence

$$\begin{aligned}
&D_n(\{p_1, p_2, \cdots, p_n\}; \{q_1, q_2, \cdots, q_n\}; \lambda) \\
&= (\lambda - p_n) D_{n-1}(\{p_1, p_2, \cdots, p_{n-1}\}; \{q_1, q_2, \cdots, q_{n-1}\}; \lambda) \\
&\quad + q_n^2 (\lambda - p_1)(\lambda - p_2) \cdots (\lambda - p_{n-1}).
\end{aligned} \tag{2.4}$$

We will deduce from (2.4), by induction, that for $n \geq 2$,

$$D_n(\{p_1, p_2, \cdots, p_n\}; \{q_1, q_2, \cdots, q_n\}; \lambda) = \prod_{i=1}^{n} (\lambda - p_i) + \sum_{j=1}^{n} \left(q_j^2 \prod_{i \neq j} (\lambda - p_i) \right). \tag{2.5}$$

For $n = 2$,

$$\begin{aligned}
D_2(\{p_1, p_2\}; \{q_1, q_2\}; \lambda) &= \begin{vmatrix} \lambda - p_1 + q_1^2 & q_1 q_2 \\ q_1 q_2 & \lambda - p_2 + q_2^2 \end{vmatrix} \\
&= (\lambda - p_1)(\lambda - p_2) + q_1^2 (\lambda - p_2) + q_2^2 (\lambda - p_1).
\end{aligned}$$

That is, (2.5) holds for $n = 2$. We now assume (2.5) holds for $n - 1 \geq 2$. Then by (2.4) and the induction hypothesis,

$$\begin{aligned}
&D_n(\{p_1, p_2, \cdots, p_n\}; \{q_1, q_2, \cdots, q_n\}; \lambda) \\
&= (\lambda - p_n) D_{n-1}(\{p_1, p_2, \cdots, p_{n-1}\}; \{q_1, q_2, \cdots, q_{n-1}\}; \lambda) \\
&\quad + q_n^2 (\lambda - p_1)(\lambda - p_2) \cdots (\lambda - p_{n-1}) \\
&= (\lambda - p_n) \left(\prod_{i=1}^{n-1} (\lambda - p_i) + \sum_{j=1}^{n-1} \left(q_j^2 \prod_{i \neq j, i \leq n-1} (\lambda - p_i) \right) \right) \\
&\quad + q_n^2 (\lambda - p_1)(\lambda - p_2) \cdots (\lambda - p_{n-1}) \\
&= \prod_{i=1}^{n} (\lambda - p_i) + \sum_{j=1}^{n} \left(q_j^2 \prod_{i \neq j} (\lambda - p_i) \right).
\end{aligned}$$

We have proved that (2.5) holds for $n \geq 2$. Recall the Veite theorem that for any $n \times n$ matrix U,

$$\det(\lambda I - U) = \sum_{i=0}^{n} (-1)^i \sigma_i(\lambda(U)) \lambda^{n-i}. \tag{2.6}$$

In particular, if $U = \mathrm{diag}(p_1, p_2 \cdots, p_2)$,

$$\prod_{i=1}^{n} (\lambda - p_i) = \sum_{i=0}^{n} (-1)^i \sigma_i(p) \lambda^{n-i}, \tag{2.7}$$

here $p = (p_1, p_2 \cdots, p_n)$. Using (2.3) and (2.7), (2.5) is written as

$$\begin{aligned}
\det(\lambda I - M) &= \sum_{i=0}^{n} (-1)^i \sigma_i(p) \lambda^{n-i} + \sum_{j=1}^{n} \left(q_j^2 \sum_{i=1}^{n} (-1)^{i-1} \sigma_{i-1;j}(p) \lambda^{n-i} \right) \\
&= \sum_{i=0}^{n} (-1)^i \left(\sigma_i(p) - \sum_{j=1}^{n} q_j^2 \sigma_{i-1;j}(p) \right) \lambda^{n-i},
\end{aligned}$$

here we used standard conventions that $\sigma_0(p) = 1$ and $\sigma_{-1}(p) = 0$. Thus, (1.7) follows from (2.6). The proof of Proposition 1.2 is completed. $\square$

Proof of Lemma 1.3. For any $A = \text{diag}(a_1, a_2, \cdots, a_n)$, if $\omega \in C^2(\mathbb{R}^n)$ is a generalized symmetric function with respect to A, that is

$$\omega(x) = \omega\left(\frac{1}{2}\sum_{i=1}^{n} a_i x_i^2\right),$$

then

$$D_i\omega(x) = \omega'(s)a_i x_i,$$
$$D_{ij}\omega(x) = \omega'(s)a_i\delta_{ij} + \omega''(s)(a_i x_i)(a_j x_j). \tag{2.8}$$

Comparing (1.6) and (2.8), letting $\beta = -\omega''(s)$, $p_i = \omega'(s)a_i$ and $q_i = a_i x_i$, and substituting them into (1.7), we have (1.8). $\square$

Symmetric solutions. For $A = c^* I$ and $2 \le k \le n$,

$$\overline{\omega}_k(s) = \int_1^s \left(1 + \alpha\, t^{-\frac{n}{2}}\right)^{\frac{1}{k}} dt, \quad \alpha > 0,\ s > 0, \tag{2.9}$$

satisfies the ordinary differential equation

$$\sigma_k(\lambda(D^2\omega)) = (\omega'(s))^k + 2s\frac{k}{n}\omega''(s)(\omega'(s))^{k-1} = 1, \quad s > 0. \tag{2.10}$$

Therefore, $\overline{\omega}_k\left(\frac{c^*}{2}|x|^2\right)$ is a solution of (1.1) in $\mathbb{R}^n \setminus \{0\}$. In order to prove Proposition 1.4, for every $a = (a_1, a_2, \cdots, a_n) \in \Gamma^+$, we denote

$$A_k^i(a) = a_i\sigma_{k-1;i}(a), \quad i = 1, 2, \cdots, n. \tag{2.11}$$

From the property of σ_k, (2.2), we have

$$\sum_{i=1}^{n} A_k^i(a) = k\sigma_k(a). \tag{2.12}$$

Proof of Proposition 1.4. To better illustrate the idea of the proof , we start with $k = 1$. For $s \in (\alpha, \beta)$, $1 \le i \le n$, let $x = (0, \cdots, 0, \sqrt{\frac{2s}{a_i}}, 0, \cdots, 0)$. We have, using $A \in \mathcal{A}_1$,

$$1 = \Delta\omega(x) = \omega'(s)\sum_{j=1}^{n} a_j + \omega''(s)\sum_{j=1}^{n} a_j^2 x_j^2 = \omega'(s) + 2s\omega''(s)a_i.$$

Since $\omega'' \not\equiv 0$ in (α, β), there exists some $\overline{s} \in (\alpha, \beta)$ such that $\omega''(\overline{s}) \neq 0$. It follows that

$$a_i = \frac{1 - \omega'(\overline{s})}{2\overline{s}\omega''(\overline{s})}$$

is independent of i. Since $A \in \mathcal{A}_1$, $1 = \sum_{i=1}^{n} a_i$. So $a_1 = a_2 = \cdots = a_n = \frac{1}{n}$. Proposition 1.4 for $k = 1$ is established.

Now we consider the case $2 \leq k \leq n$. For $s \in (\alpha,\beta)$, $1 \leq i \leq n$, let $x = (0,\cdots,0,\sqrt{\frac{2s}{a_i}},0,\cdots,0)$, we have, using Lemma 1.3,

$$\begin{aligned}
1 &= \sigma_k(\lambda(D^2\omega(x))) \\
&= \sigma_k(a)(\omega'(s))^k + \omega''(s)(\omega'(s))^{k-1}\sigma_{k-1;j}(a)(a_jx_j)^2 \\
&= (\omega'(s))^k + 2s\omega''(s)(\omega'(s))^{k-1}\sigma_{k-1;i}(a)a_i.
\end{aligned}$$

It is clear from the above that $\omega'(s) \neq 0$, $\forall\ s \in (\alpha,\beta)$. Since $\omega'' \not\equiv 0$ in (α,β), there exists some $\bar{s} \in (\alpha,\beta)$ such that $\omega''(\bar{s}) \neq 0$. It follows that

$$A_k^i(a) = \sigma_{k-1;i}(a)a_i = \frac{1-(\omega'(\bar{s}))^k}{2\bar{s}\omega''(\bar{s})(\omega'(\bar{s}))^{k-1}}$$

is independent of i. For $2 \leq k \leq n-1$, for any $i_1, i_2 \in \{1,2,\cdots,n\}$, by (2.11) and (2.1), we have

$$\begin{aligned}
0 &= A_k^{i_1}(a) - A_k^{i_2}(a) \\
&= a_{i_1}\sigma_{k-1;i_1}(a) - a_{i_2}\sigma_{k-1;i_2}(a) \\
&= a_{i_1}\left(a_{i_2}\sigma_{k-2;i_1i_2}(a) + \sigma_{k-1;i_1i_2}(a)\right) - a_{i_2}\left(a_{i_1}\sigma_{k-2;i_1i_2}(a) + \sigma_{k-1;i_1i_2}(a)\right) \\
&= (a_{i_1} - a_{i_2})\sigma_{k-1;i_1i_2}(a).
\end{aligned} \tag{2.13}$$

Since $a_i > 0$, $i = 1,2,\cdots,n$, it follows that $\sigma_{k-1;i_1i_2}(a) \neq 0$. By the arbitrariness of i_1, i_2, we have $a_1 = a_2 = \cdots = a_n$. Using $\sigma_k(a) = 1$, we have

$$a_1 = a_2 = \cdots = a_n = (C_n^k)^{-1/k}.$$

Proposition 1.4 is proved. □

Generalized symmetric subsolutions. From Proposition 1.4, we see that there is no generalized symmetric solutions of (1.1) with $\omega''(s) \not\equiv 0$ in remaining cases. We will construct a family of generalized symmetric smooth functions satisfying

$$\omega'(s) > 0, \quad \omega''(s) \leq 0,$$

and

$$\sigma_k(\lambda(D^2\omega)) \geq 1, \quad \text{and} \quad \sigma_m(\lambda(D^2\omega)) \geq 0,\ 1 \leq m \leq k-1.$$

For $A = \mathrm{diag}(a_1,a_2,\cdots,a_n) \in \mathcal{A}_k$, denote $a = (a_1,a_2,\cdots,a_n)$, and consider

$$h_k(a) := \max_{1\leq i\leq n} A_k^i(a). \tag{2.14}$$

Since $A_n^i(a) = a_i\sigma_{n-1;i}(a) = \sigma_n(a)$ for every i, we have $h_n(a) = 1$. By (2.11), (2.1) and (2.12), we have, for $1 \leq k \leq n-1$,

$$A_k^i(a) = a_i\sigma_{k-1;i}(a) < \sigma_k(a) = 1, \quad \forall\ i,$$

and

$$nh_k(a) \geq \sum_{i=1}^{n} A_k^i(a) = k\sigma_k(a) = k.$$

We see from the above that

$$\frac{k}{n} \leq h_k(a) < 1, \tag{2.15}$$

with " = " holds if and only if $A_k^i(a)$ is independent of i, i.e., in view of (2.13), $a_1 = a_2 = \cdots = a_n = c^*$. For $n \geq 3$ and $2 \leq k \leq n$, in view of (2.15) and $h_n(a) = 1$, we have

$$\frac{k}{2h_k(a)} > 1. \tag{2.16}$$

By a simple computation, the following ordinary differential equation

$$\begin{cases} (\omega'(s))^k + 2h_k(a)s\omega''(s)(\omega'(s))^{k-1} = 1, & s > 0, \\ \omega'(s) > 0, \quad \omega''(s) \leq 0 \end{cases} \tag{2.17}$$

has a family of solutions

$$\omega_\alpha(s) = \beta + \int_{\bar{s}}^{s} \left(1 + \alpha\, t^{-\frac{k}{2h_k(a)}}\right)^{\frac{1}{k}} dt, \quad \alpha > 0,\ s > 0, \tag{2.18}$$

where $\beta \in \mathbb{R}$ and $\bar{s} > 0$. It follows from (2.16) that

$$\begin{aligned} \omega_\alpha(s) &= \beta + s - \bar{s} + \int_{\bar{s}}^{s} \left(\left(1 + \alpha\, t^{-\frac{k}{2h_k(a)}}\right)^{\frac{1}{k}} - 1\right) dt \\ &= s + \mu(\alpha) + O\left(s^{\frac{(2-n)\theta}{2}}\right), \quad \text{as} \quad s \to \infty, \end{aligned} \tag{2.19}$$

where

$$\mu(\alpha) = \beta - \bar{s} + \int_{\bar{s}}^{\infty} \left(\left(1 + \alpha\, t^{-\frac{k}{2h_k(a)}}\right)^{\frac{1}{k}} - 1\right) dt < \infty,$$

and

$$\theta = \frac{1}{n-2}\left(\frac{k}{h_k(a)} - 2\right).$$

We see from (2.15) that $\theta \in \left(\frac{k-2}{n-2}, 1\right]$ if $2 \leq k \leq n-1$, and $\theta = 1$ if $k = n$.

Proposition 2.1. *For $n \geq 3$ and $2 \leq k \leq n$, $A \in \mathcal{A}_k$, let $\omega_\alpha(x) = \omega_\alpha\left(\frac{1}{2}x^T Ax\right)$ be given in* (2.18)*. Then ω_α is a smooth k-convex subsolution of* (1.1) *in $\mathbb{R}^n \setminus \{0\}$ satisfying*

$$\omega_\alpha(x) = \frac{1}{2}x^T Ax + \mu(\alpha) + O\left(|x|^{\theta(2-n)}\right), \qquad \textit{as} \quad x \to \infty. \tag{2.20}$$

Proof. Obviously, (2.20) follows from (2.19). By computation,

$$\omega_\alpha'(s) = \left(1 + \alpha\, s^{-\frac{k}{2h_k(a)}}\right)^{\frac{1}{k}} > 1,$$

$$\omega_\alpha''(s) = -\frac{1}{2h_k(a)s} \cdot \frac{\alpha}{s^{\frac{k}{2h_k(a)}} + \alpha} \cdot \omega_\alpha'(s) < 0. \tag{2.21}$$

It is clear from Lemma 1.3, (2.14) and (2.17) that

$$\sigma_k(\lambda(D^2u)) \geq \sigma_k(a)(\omega_\alpha')^k + h_k(a)\omega_\alpha''(\omega_\alpha')^{k-1}2s = 1, \quad \text{in } \mathbb{R}^n \setminus \{0\}.$$

By Lemma 1.3, (2.21) and (2.14), we have, for any $1 \le m \le k-1$,

$$\begin{aligned}
\sigma_m(\lambda(D^2u)) &= \sigma_m(a)(\omega_\alpha')^m + \omega_\alpha''(\omega_\alpha')^{m-1}\sum_{i=1}^n \sigma_{m-1;i}(a)(a_ix_i)^2 \\
&= (\omega_\alpha')^m\left(\sigma_m(a) - \frac{1}{2sh_k(a)}\cdot\frac{\alpha}{s^{\frac{k}{2h_k(a)}}+\alpha}\sum_{i=1}^n \sigma_{m-1;i}(a)(a_ix_i)^2\right) \\
&\ge (\omega_\alpha')^m\left(\sigma_m(a) - \frac{1}{2s}\cdot\frac{\alpha}{s^{\frac{k}{2h_k(a)}}+\alpha}\sum_{i=1}^n \frac{\sigma_{m-1;i}(a)(a_ix_i)^2}{a_i\sigma_{k-1;i}(a)}\right).
\end{aligned}$$

In order to show $\sigma_m(\lambda(D^2u)) \ge 0$, it suffices to prove, for each $1 \le i \le n$,

$$\sigma_m(a)\sigma_{k-1;i}(a) \ge \sigma_{m-1;i}(a). \tag{2.22}$$

Note that the Newtonian inequalities may be expressed as

$$\frac{\sigma_{k+1}(a)}{C_n^{k+1}}\cdot\frac{\sigma_{k-1}(a)}{C_n^{k-1}} \le \left(\frac{\sigma_k(a)}{C_n^k}\right)^2,$$

for $1 \le k \le n-1$. Since

$$\frac{C_n^{k-1}C_n^{k+1}}{C_n^kC_n^k} = \frac{(n-k)k}{(n-k+1)(k+1)} < 1,$$

it follows that

$$\frac{\sigma_{k+1}(a)}{\sigma_k(a)} \le \frac{\sigma_k(a)}{\sigma_{k-1}(a)},$$

which shows that the Hessian quotient $\frac{\sigma_{k+1}(a)}{\sigma_k(a)}$ is decreasing with respect to k. So we have for any $m \le k$, and each $1 \le i \le n$,

$$\sigma_{m;i}(a)\sigma_{k-1;i}(a) \ge \sigma_{m-1;i}(a)\sigma_{k;i}(a),$$

Then by the property (2.1), it follows that

$$\begin{aligned}
\sigma_m(a)\sigma_{k-1;i}(a) &= (\sigma_{m;i}(a) + a_i\sigma_{m-1;i}(a))\,\sigma_{k-1;i}(a) \\
&\ge \sigma_{m-1;i}(a)\cdot\sigma_{k;i}(a) + \sigma_{m-1;i}(a)\cdot\, a_i\sigma_{k-1;i}(a) \\
&= \sigma_{m-1;i}(a)\sigma_k(a) \\
&= \sigma_{m-1;i}(a).
\end{aligned}$$

i.e. (2.22) is proved. Hence ω_α is a smooth k-convex subsolution of (1.1) in $\mathbb{R}^n\backslash\{0\}$. □

3 Proof of Theorem 1.1

The following Lemma holds for any invertible and symmetric matrix A, and A is not necessarily diagonal or in $\mathcal{A}_k$, $2 \le k \le n$.

Lemma 3.1. *Let $\varphi \in C^2(\partial D)$. There exists some constant C, depending only on n, $\|\varphi\|_{C^2(\partial D)}$, the upper bound of A, the diameter and the convexity of D, and the C^2 norm of ∂D, such that, for every $\xi \in \partial D$, there exists $\bar{x}(\xi) \in \mathbb{R}^n$ satisfying*

$$|\bar{x}(\xi)| \leq C \quad \textit{and} \quad w_\xi < \varphi \textit{ on } \overline{D} \setminus \{\xi\},$$

where

$$w_\xi(x) := \varphi(\xi) + \frac{1}{2}\left((x - \bar{x}(\xi))^T A(x - \bar{x}(\xi)) - (\xi - \bar{x}(\xi))^T A(\xi - \bar{x}(\xi))\right), \quad x \in \mathbb{R}^n.$$

Proof. Let $\xi \in \partial D$. By a translation and a rotation, we may assume without loss of generality that $\xi = 0$ and ∂D is locally represented by the graph of

$$x_n = \rho(x') = O(|x'|^2),$$

and φ locally has the expansion

$$\begin{aligned}\varphi(x', \rho(x')) &= \varphi(0) + \varphi_{x_1}(0)x_1 + \cdots + \varphi_{x_n}(0)x_n + O(|x|^2)\\ &= \varphi(0) + \varphi_{x_1}(0)x_1 + \cdots + \varphi_{x_{n-1}}(0)x_{n-1} + O(|x'|^2),\end{aligned}$$

where $x' = (x_1, \cdots, x_{n-1})$.

Since A is invertible, we can find $\bar{x} = \bar{x}(t) \in \mathbb{R}^n$ such that, for appropriate t to fit our need later,

$$A\bar{x}(t) = (-\varphi_{x_1}(0), \cdots, -\varphi_{x_{n-1}}(0), t)^T.$$

Let

$$w(x) = \varphi(0) + \frac{1}{2}\left((x - \bar{x})^T A(x - \bar{x}) - \bar{x}^T A\bar{x}\right), \quad x \in \mathbb{R}^n.$$

Then

$$w(x) = \varphi(0) + \frac{1}{2}x^T Ax - x^T A\bar{x} = \varphi(0) + \frac{1}{2}x^T Ax + \sum_{\alpha=1}^{n-1} \varphi_{x_\alpha}(0)x_\alpha - tx_n. \tag{3.1}$$

It follows that

$$\begin{aligned}(w - \varphi)(x', \rho(x')) &= \frac{1}{2}x^T Ax - t\rho(x') + O(|x'|^2)\\ &\leq C\left(|x'|^2 + \rho(x')^2\right) - t\rho(x'),\end{aligned}$$

where C depends only on the upper bound of A, $\|\varphi\|_{C^2(\partial D)}$, and the C^2 norm of ∂D. By the strict convexity of ∂D, there exists some constant $\delta > 0$ depending only on D such that

$$\rho(x') \geq \delta|x'|^2, \quad \forall\ |x'| < \delta. \tag{3.2}$$

Clearly, for large t, we have

$$(w - \varphi)(x', \rho(x')) < 0, \quad \forall\ 0 < |x'| < \delta.$$

The largeness of t depends only on $\delta, A, \|\varphi\|_{C^2(\partial D)}$, and the C^2 norm of ∂D.

On the other hand, by the strict convexity of ∂D and (3.2),

$$x_n \geq \delta^3, \quad \forall\, x \in \partial D \setminus \{(x', \rho(x')) \bigm| |x'| < \delta\}.$$

It follows from (3.1) that

$$w(x) \leq\ C - \delta^3 t, \quad \forall\, x \in \partial D \setminus \{(x', \rho(x')) \bigm| |x'| < \delta\},$$

where C depends only on A, diam(D), $\|\varphi\|_{C^2(\partial D)}$. By making t large (still under control), we have

$$w(x) - \varphi(x) < 0, \quad \forall\, x \in \partial D \setminus \{(x', \rho(x')) \bigm| |x'| < \delta\}.$$

Lemma 3.1 is established. □

By an orthogonal transformation and by subtracting a linear function from u, we only need to prove Theorem 1.1 for the case that $A = \mathrm{diag}(a_1, a_2, \cdots, a_n)$ where $a_i > 0$ $(1 \leq i \leq n)$, $b = 0$.

Proof of Theorem 1.1. Without loss of generality, we assume that $0 \in D$. For $s > 0$, let

$$E(s) := \left\{x \in \mathbb{R}^n \Bigm| \frac{1}{2} x^T A x < s\right\}.$$

Fix $\bar{s} > 0$ such that $\overline{D} \subset E(\bar{s})$. For $\alpha > 0, \beta \in \mathbb{R}$, set

$$\omega_\alpha(x) = \beta + \int_{\bar{s}}^{\frac{1}{2}x^T A x} \left(1 + \alpha t^{-\frac{k}{2h_k(a)}}\right)^{\frac{1}{k}} dt,$$

as in (2.18). We have by Proposition 2.1 that ω_α is a smooth k-convex subsolution of (1.1) in $\mathbb{R}^n \setminus \{0\}$, and

$$\omega_\alpha(x) = \frac{1}{2} x^T A x + \mu(\alpha) + O\left(|x|^{\theta(2-n)}\right), \qquad \text{as} \quad x \to \infty.$$

Here

$$\mu(\alpha) = \beta - \bar{s} + \int_{\bar{s}}^{\infty} \left(\left(1 + \alpha\, t^{-\frac{k}{2h_k(a)}}\right)^{\frac{1}{k}} - 1\right) dt, \qquad \theta \in \left[\frac{k-2}{n-2}, 1\right].$$

Clearly, $\mu(\alpha)$ is strictly increasing in α, and

$$\lim_{\alpha \to \infty} \mu(\alpha) = \infty. \tag{3.3}$$

On the other hand,

$$\omega_\alpha \leq \beta, \qquad \text{in } E(\bar{s}) \setminus \overline{D},\ \forall\, \alpha > 0. \tag{3.4}$$

Let

$$\beta := \min\left\{w_\xi(x) \bigm| \xi \in \partial D,\ x \in \overline{E(\bar{s})} \setminus D\right\},$$
$$\widehat{b} := \max\left\{w_\xi(x) \bigm| \xi \in \partial D,\ x \in \overline{E(\bar{s})} \setminus D\right\},$$

where $w_\xi(x)$ is given by Lemma 3.1. We will fix the value of c_* in the proof. First we require that c_* satisfies $c_* > \widehat{b}$. It follows that

$$\mu(0) = \beta - \bar{s} < \beta \leq \widehat{b} < c_*.$$

Thus, in view of (3.3), for every $c > c_*$, there exists a unique $\alpha(c)$ such that

$$\mu(\alpha(c)) = c. \tag{3.5}$$

So $\omega_{\alpha(c)}$ satisfies

$$\omega_{\alpha(c)}(x) = \frac{1}{2}x^T Ax + c + O\left(|x|^{\theta(2-n)}\right), \qquad \text{as} \quad x \to \infty. \tag{3.6}$$

Set

$$\underline{w}(x) = \max\left\{w_\xi(x) \,\middle|\, \xi \in \partial D\right\}.$$

It is clear by Lemma 3.1 that $\underline{w}$ is a locally Lipschitz function in $\mathbb{R}^n \setminus D$, and $\underline{w} = \varphi$ on ∂D. Since w_ξ is a smooth convex solution of (1.1), $\underline{w}$ is a viscosity subsolution of (1.1) in $\mathbb{R}^n \setminus \overline{D}$. We fix a number $\hat{s} > \bar{s}$, and then choose another number $\widehat{\alpha} > 0$ such that

$$\min_{\partial E(\hat{s})} \omega_{\widehat{\alpha}} > \max_{\partial E(\hat{s})} \underline{w}.$$

We require that c_* also satisfies $c_* \geq \mu(\widehat{\alpha})$. We now fix the value of c_*.

For $c \geq c_*$, we have $\alpha(c) = \mu^{-1}(c) \geq \mu^{-1}(c_*) \geq \widehat{\alpha}$, and therefore

$$\omega_{\alpha(c)} \geq \omega_{\widehat{\alpha}} > \underline{w}, \quad \text{on } \partial E(\hat{s}). \tag{3.7}$$

By (3.4), we have

$$\omega_{\alpha(c)} \leq \beta \leq \underline{w}, \quad \text{in } E(\bar{s}) \setminus \overline{D}. \tag{3.8}$$

Now we define, for $c > c_*$,

$$\underline{u}(x) = \begin{cases} \max\left\{\omega_{\alpha(c)}(x), \underline{w}(x)\right\}, & x \in E(\hat{s}) \setminus D, \\ \omega_{\alpha(c)}(x), & x \in \mathbb{R}^n \setminus E(\hat{s}). \end{cases}$$

We know from (3.8) that

$$\underline{u} = \underline{w}, \quad \text{in } E(\bar{s}) \setminus \overline{D}, \tag{3.9}$$

and in particular

$$\underline{u} = \underline{w} = \varphi, \quad \text{on } \partial D. \tag{3.10}$$

We know from (3.7) that $\underline{u} = \omega_{\alpha(c)}$ in a neighborhood of $\partial E(\hat{s})$. Therefore $\underline{u}$ is locally Lipschitz in $\mathbb{R}^n \setminus D$. Since both $\omega_{\alpha(c)}$ and $\underline{w}$ are viscosity subsolutions of (1.1) in $\mathbb{R}^n \setminus \overline{D}$, so is $\underline{u}$.

For $c > c_*$,

$$\overline{u}(x) := \frac{1}{2}x^T Ax + c$$

is a smooth convex solution of (1.1). By (3.8),

$$\omega_{\alpha(c)} \leq \beta \leq \widehat{b} < c_* < \overline{u}, \quad \text{on } \partial D.$$

We also know by (3.6) that

$$\lim_{|x|\to\infty} (\omega_{\alpha(c)}(x) - \overline{u}(x)) = 0.$$

Thus, in view of the comparison principle for smooth k-convex solutions of (1.1), (see [4]), we have

$$\omega_{\alpha(c)} \le \overline{u}, \quad \text{on } \mathbb{R}^n \setminus D. \tag{3.11}$$

By (3.7) and the above, we have, for $c > c_*$,

$$w_\xi \le \overline{u}, \quad \text{on } \partial(E(\hat{s}) \setminus D), \ \forall\, \xi \in \partial D.$$

By the comparison principle for smooth convex solutions of (1.1), we have

$$w_\xi \le \overline{u}, \quad \text{in } E(\hat{s}) \setminus \overline{D}, \ \forall\, \xi \in \partial D.$$

Thus

$$\underline{w} \le \overline{u}, \quad \text{in } E(\hat{s}) \setminus \overline{D}.$$

This, combining with (3.11), implies that

$$\underline{u} \le \overline{u}, \quad \text{in } \mathbb{R}^n \setminus D.$$

For any $c > c_*$, let $\mathcal{S}_c$ denote the set of $v \in \mathrm{USC}(\mathbb{R}^n \setminus D)$ which are viscosity subsolutions of (1.1) in $\mathbb{R}^n \setminus \overline{D}$ satisfying

$$v = \varphi, \quad \text{on } \partial D, \tag{3.12}$$

and

$$\underline{u} \le v \le \overline{u}, \quad \text{in } \mathbb{R}^n \setminus D. \tag{3.13}$$

We know that $\underline{u} \in \mathcal{S}_c$. Let

$$u(x) := \sup \{v(x) \mid v \in \mathcal{S}_c\}, \quad x \in \mathbb{R}^n \setminus D.$$

By (3.6), and the definitions of $\underline{u}$ and $\overline{u}$,

$$u(x) \ge \underline{u}(x) = \omega_{\alpha(c)}(x) = \frac{1}{2}x^T Ax + c + O\left(|x|^{\theta(2-n)}\right), \qquad \text{as} \quad x \to \infty. \tag{3.14}$$

and

$$u(x) \le \overline{u}(x) = \frac{1}{2}x^T Ax + c.$$

The estimate (1.4) follows.

Next, we prove that u satisfies the boundary condition. It is obvious from (3.10) that

$$\liminf_{x\to\xi} u(x) \ge \lim_{x\to\xi} \underline{u}(x) = \varphi(\xi), \quad \forall\, \xi \in \partial D.$$

So we only need to prove that

$$\limsup_{x\to\xi} u(x) \le \varphi(\xi), \quad \forall\, \xi \in \partial D.$$

Let $\omega_c^+ \in C^2(\overline{E(\overline{s}) \setminus D})$ be defined by

$$\begin{cases} \Delta\omega_c^+ = 0, & \text{in } E(\overline{s}) \setminus \overline{D}, \\ \omega_c^+ = \varphi, & \text{on } \partial D, \\ \omega_c^+ = \max\limits_{\partial E(\overline{s})} \overline{u} = \overline{s} + c, & \text{on } \partial E(\overline{s}). \end{cases}$$

It is easy to see that a viscosity subsolution v of (1.1) satisfies $\Delta v \geq 0$ in viscosity sense. Therefore, for every $v \in \mathcal{S}_c$, by $v \leq \omega_c^+$ on $\partial(E(\bar{s}) \setminus D)$, we have

$$v \leq \omega_c^+ \quad \text{in } E(\bar{s}) \setminus \overline{D}.$$

It follows that

$$u \leq \omega_c^+ \quad \text{in } E(\bar{s}) \setminus \overline{D},$$

and then

$$\limsup_{x \to \xi} u(x) \leq \lim_{x \to \xi} \omega_c^+(x) = \varphi(\xi), \quad \forall\, \xi \in \partial D.$$

Finally, we prove that u is a viscosity solution of (1.1). The following ingredients for the viscosity adaptation of Perron's method (see [14]) are available.

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $u \in \mathrm{LSC}(\overline{\Omega})$ and $v \in \mathrm{USC}(\overline{\Omega})$ are respectively viscosity supersolutions and subsolutions of* (1.1) *in Ω satisfying $u \geq v$ on $\partial\Omega$. Then $u \geq v$ in Ω.*

Under the assumptions $u, v \in C^0(\overline{\Omega})$, the lemma was proved in [25], based on Jensen approximations (see [15]). The proof remains valid under the weaker regularity assumptions on u and v.

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $\mathcal{S}$ be a non-empty family of viscosity subsolutions (supersolutions) of* (1.1) *in Ω. Set*

$$u(x) = \sup\ (\inf)\ \{v(x) \mid v \in \mathcal{S}\},$$

and

$$u^*\ (u_*)\ (x) = \lim_{r \to 0} \sup_{B_r}\ (\inf_{B_r})\ u$$

be the upper (lower) semicontinuous envelope of u. Then, if $u^* < \infty$ ($u_* > -\infty$) in Ω, u^* (u_*) is a viscosity subsolution (supersolution) of (1.1) *in Ω.*

Lemma 3.3 can be proved by standard arguments, see e.g. [8]. With these ingredients, an application of the Perron process, see e.g. Lemma 4.4 in [8], gives that $u \in C^0(\mathbb{R}^n \setminus D)$ is a viscosity solution of (1.3). Theorem 1.1 is established. □

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